Convergence of the empirical measure induced by a Moran type particle system

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Motivation

- **Problem** approach ν : **quasi-stationary distribution (QSD)** of an absorbing MC $(X_t)_{t\geq 0}$ with transitions

$$x \xrightarrow{\mu_{x,y}} y \quad \text{et} \quad x \xrightarrow{\kappa(x)} \partial \text{ (absorbing state)}$$

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$$\boxed{\nu(\cdot) := \lim_{t\to\infty} \mathbb{P}[X_t \in \cdot \mid X_t \neq \partial] \quad \text{and} \quad \mathbb{P}[X_t \neq \partial] \to 0}$$

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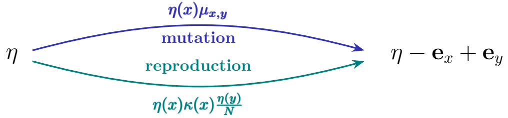
- **Fleming – Viot particle system:**

$\eta = (\eta(1), \dots, \eta(x), \dots)$, where $\eta(x) =$ nb. of particles of the type x, and $|\eta| = N$



Particle system:

$(\eta_t^{(N)})_{t\geq 0}$

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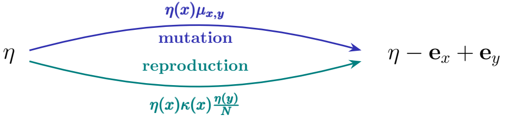

Particle system:
$(\eta_t^{(N)})_{t\geq 0}$

- **Empirical measure:** $m(\eta_t^{(N)}) = \sum_{x\in E} \frac{\eta_t^{(N)}(x)}{N} \delta_x$

The empirical measure $m(\eta_t^{(N)})$ approaches $\mathbb{P}[X_t \in \cdot \mid X_t \neq \partial]$ when $N \to \infty$

- **Fleming – Viot particle system:**

$$\eta \xrightarrow[\text{reproduction}]{\text{mutation}} \eta - \mathbf{e}_x + \mathbf{e}_y$$

mutation rate: $\eta(x)\mu_{x,y}$; reproduction rate: $\eta(x)\kappa(x)\frac{\eta(y)}{N}$

- **Multi-allelic Moran process:**

$$\eta \xrightarrow[\text{reproduction}]{\text{mutation}} \eta - \mathbf{e}_x + \mathbf{e}_y$$

mutation rate: $\eta(x)\mu_{x,y}$; reproduction rate: $\eta(x)V_{m(\eta)}(x,y)\frac{\eta(y)}{N}$

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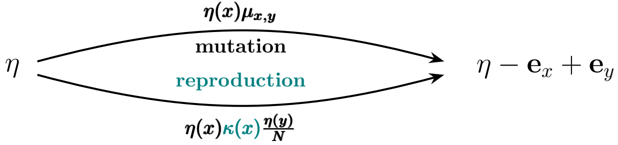

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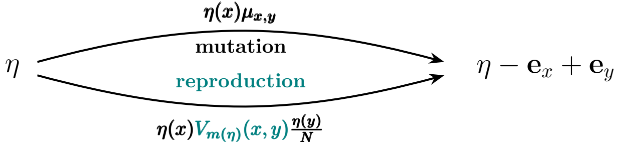


Questions

- ➥ For which selection rates does the Moran process approach a QSD?
- ➥ Speed of convergence when $N \to \infty$?
- ➥ What is the "optimal" selection rate for approaching a QSD?

Uniform in time propagation of chaos for a Moran model **J. C.** & B. Cloez (SPA, 2022)

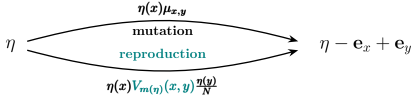
Multi-allelic Moran model

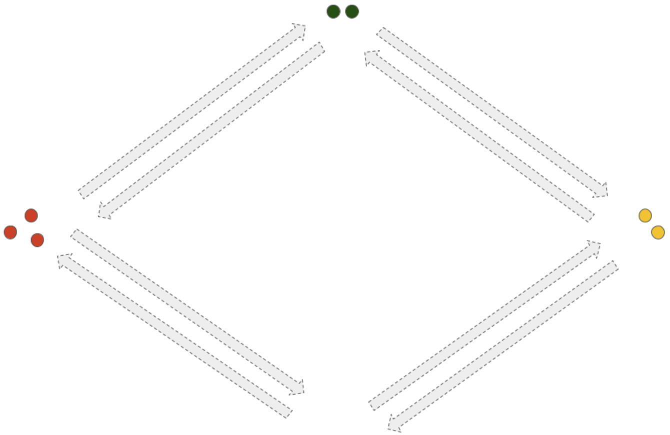
- set of possible allelic types: E (countable)
- number of individuals in the population: N
- state space of the process:

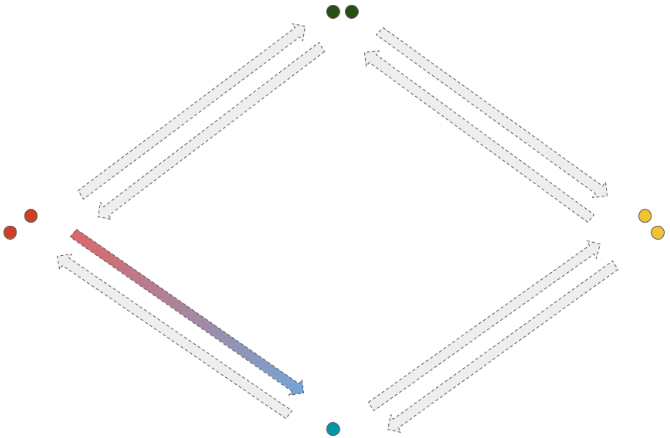
$$\mathcal{E}_N := \{\eta \in \mathbb{N}_0^E : \eta(1) + \cdots + \underbrace{\eta(k)}_{\substack{\text{nb. of indiv.}\\ \text{of type } k}} + \cdots = N\}$$

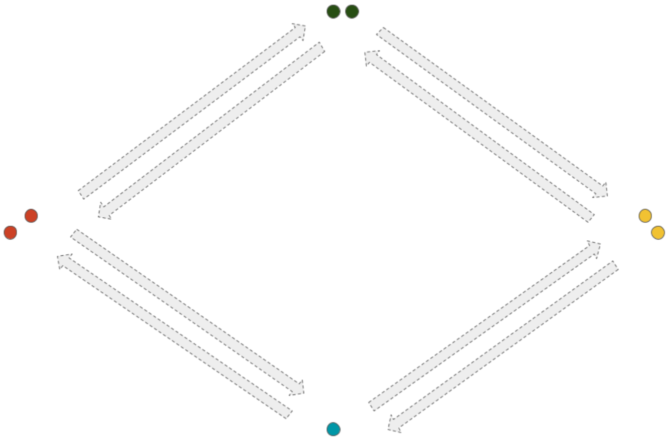
- state of the process at time t: $\eta_t^{(N)}$
- Interactions:

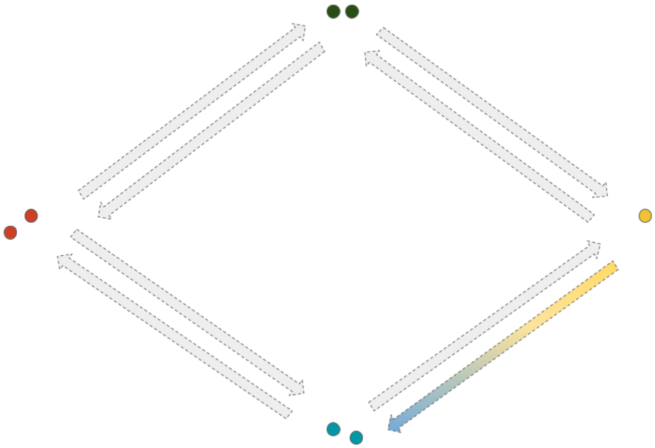
  - **mutation**: each individual mutates independently of the others according to an **irreducible** Markov chain
  - **reproduction**: one indiv. dies and another *randomly chosen* is duplicated (Moran type)

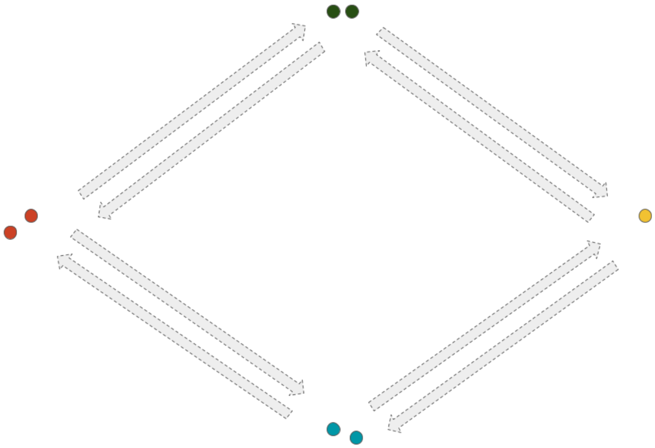


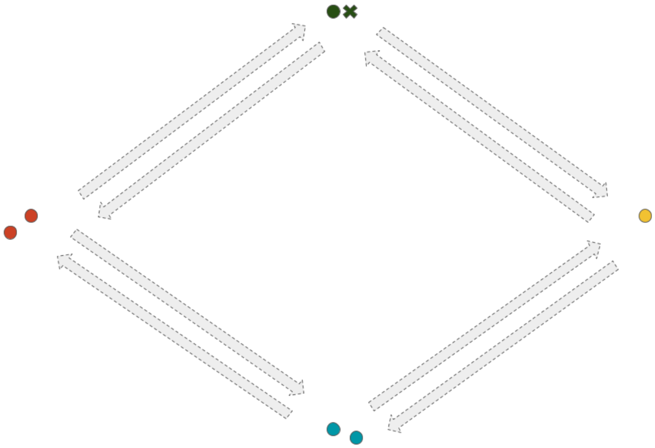


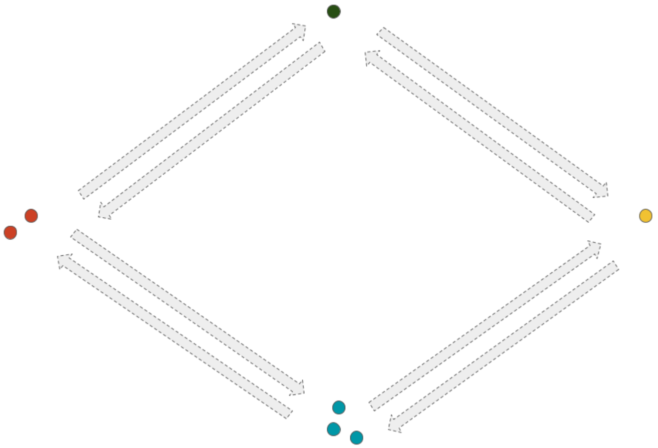


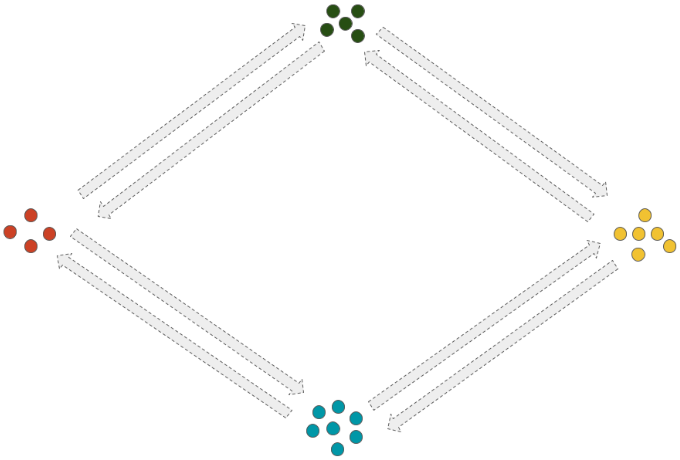












Multi-allelic Moran model

- Induced empirical distribution: $m(\eta) := \sum_{x \in E} \frac{\eta(x)}{N} \delta_x$
- Generator:

$$\mathcal{Q}_N[\eta, \eta - \mathbf{e}_x + \mathbf{e}_y] = \eta(x) \left(\underbrace{q_{x,y}}_{\text{mutation}} + \underbrace{V_{m(\eta)}(x,y)}_{\text{reproduc. rate}} \underbrace{\frac{\eta(y)}{N}}_{\text{indiv. to reproduce is of type } y} \right)$$

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Special cases:

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- selection at death: $V_\mu(x,y) = V^{\mathrm{d}}(x)$, **Fleming – Viot particle systems**
- additive selection: $V_\mu(x,y) = V^{\mathrm{d}}_\mu(x) + V^{\mathrm{b}}_\mu(y)$

  $\rightsquigarrow$ favor the indiv. with relatively high values of $\Lambda = V^{\mathrm{b}}_\mu - V^{\mathrm{d}}_\mu$

Main problems and motivations

- Study the following convergences (existence, speed, . . .)

$$\begin{array}{ccc} m\left(\eta_t^{(N)}\right) & \xrightarrow[t\to\infty]{} & m\left(\eta_\infty^{(N)}\right) \\ N\big\downarrow & & \big\downarrow N \\ \mu_t & \xrightarrow[t\to\infty]{} & \mu_\infty \end{array}$$

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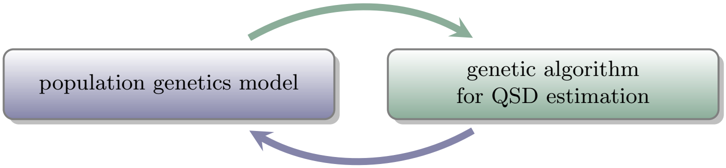

Outline

- **Additive reproduction rates**
- Propagation of chaos
- Asymptotic normality

Link to QSD

(AD) Additive selection

$$V_\mu(x,y) = V_\mu^{\mathrm{d}}(x) + V_\mu^{\mathrm{b}}(y) + V_\mu^{\mathrm{s}}(x,y), \text{ such that}$$

$V_\mu^{\mathrm{b}} - V_\mu^{\mathrm{d}} = \Lambda + C_\mu$, where Λ is bounded and does not depend on μ

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Main questions

- What is the speed of convergence?
- What system is the best for approximating a given absorbing Markov chain?

Outline

- ✓ Additive reproduction rates
- ■ **Uniform in time propagation of chaos**
- ❑ Asymptotic normality

Convergence of the empirical measure

Hypothesis (IC) Initial condition (chaos or LLN)

$$\sup_{\|\phi\|\leq 1} \mathbb{E}[|m(\eta_0^{(N)})(\phi) - \mu_0(\phi)|^p]^{1/p} \leq \frac{C}{\sqrt{N}}$$

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Hypothesis (UC) Unif. conv. to the QSD

$$\|\mu_t - \mu_\infty\|_{\mathrm{TV}} \leq C\mathrm{e}^{-\gamma t}, \text{ for all } \mu_0 \in \mathcal{M}_1(E) \text{ and } t \geq 0$$

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Theorem (Unif. in time propagation of chaos or LLN)

Assume that **(AD)**, **(IC)** and **(UC)** are verified. Then, for every $p \ge 1$,

$$\sup_{\|\phi\|\le 1} \sup_{t\ge 0} \mathbb{E}\left[|m(\eta_t^{(N)})(\phi) - \mu_t(\phi)|^p\right]^{1/p} \le \frac{C_p}{\sqrt{N}}$$

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Related works:

- P. Del Moral & L. Miclo, Séminaire de Probabilités XXXIV (2000)
- M. Rousset, SIAM J. Math. Anal. (2006)
- P. A. Ferrari & N. Marić, EJP (2007)
- D. Villemonais, ESAIM Probab. Stat. (2014)
- B. Cloez & M.-N. Thai, SPA (2016)
- N. Champagnat & D. Villemonais, ALEA (2021)

  ...

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- ✓ Additive reproduction rates
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- **Asymptotic normality**

Asymptotic normality

- Our result is a Law of Large Numbers

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- **What about a Central Limit Theorem?**

Asymptotic normality

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- **What about a Central Limit Theorem?**

Hypothesis (IC') Asymptotic normality for initial empirical distribution

$$\sqrt{N}\left(m(\eta_0^{(N)})(\phi) - \mu_0(\phi)\right) \xrightarrow[N\to\infty]{\text{Law}} \mathcal{N}\left(0, \mu_0(\phi^2)\right)$$

Theorem (Asymptotic normality)

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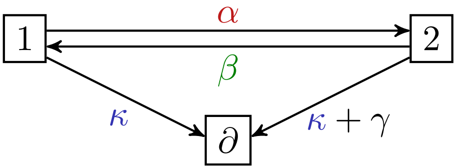
Corollary (Reduction of variance)

Take $(\eta_t^\star)_{t\geq 0}$ with selection rates $V_\mu - V_\mu^{\mathrm{s}}$. Then,

$$\lim_{N\to\infty} N \cdot \mathbb{E}\left[\left(m(\eta_T^\star)(\phi) - \mu_T(\phi)\right)^2\right] \leq \lim_{N\to\infty} N \cdot \mathbb{E}\left[\left(m(\eta_T)(\phi) - \mu_T(\phi)\right)^2\right]$$

Simplest example: two-state absorbing Markov chain

- **Problem:** estimate $\nu_\kappa(1)$: ν_κ the QSD of the absorbing Markov chain



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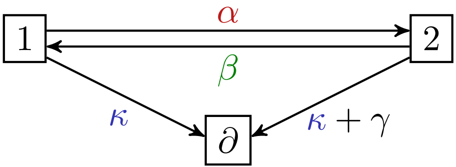

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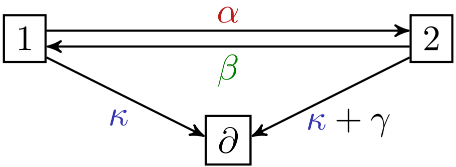

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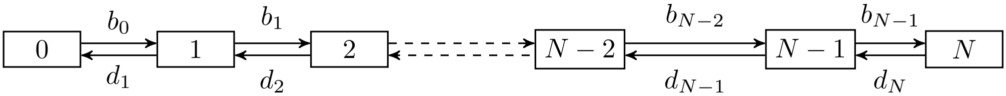

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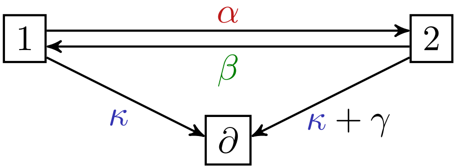

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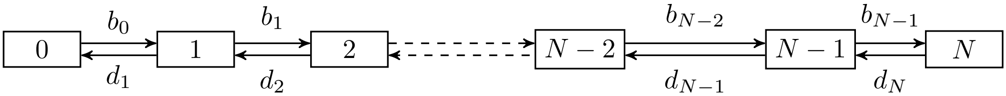

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- **Corollary:** $\mathcal{Z}^{(0)}$ estimates ν_κ with the smallest asymptotic squared error

Thank you!

What happens when $t \to \infty$?

Neutral multi-allelic Moran model

$$\mathcal{Q}_N[\eta, \eta - \mathbf{e}_x + \mathbf{e}_y] = \eta(x)\left(\underbrace{q_{x,y}}_{\text{mutation}} + \underbrace{p}_{\substack{\text{neutral rep.}\\ \text{rate}}} \underbrace{\frac{\eta(y)}{N}}_{\substack{\text{indiv. to reproduce}\\ \text{is of type } y}}\right)$$

→ explicit expression for the eigenvalues of $\mathcal{Q}_N$ in terms of the eigenvalues of Q

→ parent independent mutation $\boxed{q_{x,y} = q_y}$: reversible process with explicit stationary distribution and spectral elements

Cut-off phenomenon

Total variation distance and mixing times

Total variation distance to stationarity:

$$\mathrm{d_{TV}}(t;\eta) := \mathrm{d_{TV}}(\delta_\eta \mathrm{e}^{t\mathcal{Q}_N}, \nu_N),$$

where ν_N is the stationary distribution of the process generated by $\mathcal{Q}_N$

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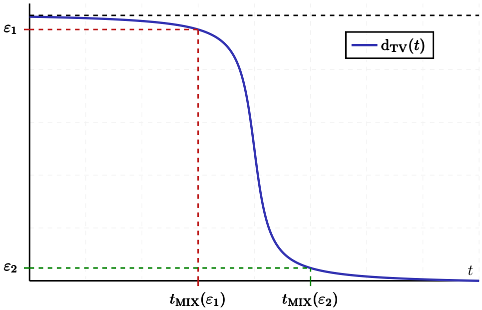
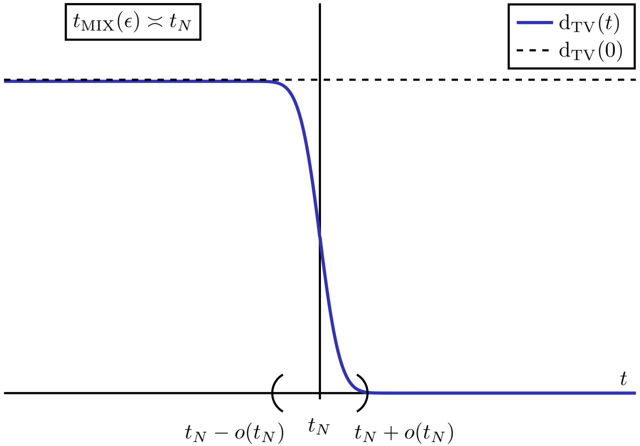
Mixing time:

$$t_{\mathrm{MIX}}(\eta;\epsilon) := \min_{t\geq 0}\{\mathrm{d_{TV}}(t;\eta) \leq \epsilon\}$$



Total variation cutoff



Diaconis (1995), Saloff-Coste (1996), Chen and Saloff-Coste (2008), Levin and Peres (2017)

Theorem (Total variation cutoff)

- **Cutoff:** for every $k \in \{1, \dots, K\}$ and $p \geq 0$, we have

$$t_{\mathrm{MIX}}(N\mathbf{e}_k, \epsilon) \asymp \frac{\ln N}{2|\boldsymbol{\mu}|}$$

  - $\left(\frac{\ln(N)}{2|\boldsymbol{\mu}|}, 1\right)$ total variation cutoff when $N \to \infty$

- **Gaussian profile when $p = 0$:**

$$\lim_{N\to\infty} \mathrm{d}^{\mathrm{TV}}_{N\mathbf{e}_k}\left(\frac{\ln N + c}{2|\boldsymbol{\mu}|}\right) = 2\Phi\left(\frac{1}{2}\sqrt{\frac{|\boldsymbol{\mu}| - \mu_k}{\mu_k}\mathrm{e}^{-c}}\right) - 1$$

On the spectrum and ergodicity of a neutral multi-allelic Moran model **J. C.** (ALEA, 2023)

Again. . .

Thank you!

Theorem (Spectrum of $\mathcal{Q}_{N,p}$)

Let $\lambda_1, \dots, \lambda_{K-1}$ be the non zero eigenvalues of Q.
Then, the non zero eigenvalues of $\mathcal{Q}_{N,p}$, counting algebraic multiplicities, are

$$\lambda_{\eta,p} = \sum_{k=1}^{K-1} \eta(k)\lambda_k - \frac{p}{N}|\eta|(|\eta| - 1), \quad \text{for any } \eta \in \bigcup_{L=1}^{N} \mathcal{E}_{K-1,L},$$

where $|\eta| := \eta(1) + \dots + \eta(K)$.

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Remarks

- Q does not need to be diagonalisable
- One also obtain some information on the eigenvectors of $\mathcal{Q}_{N,p}$

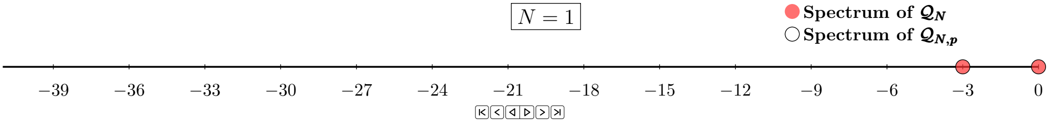
Spectrum of $\mathcal{Q}_{N,p}$ (numerical example)

Consider the mutation matrix

$$Q = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}$$

The eigenvalues of Q are $\lambda_0 = 0$ and $\lambda_1 = -3$

➜ Eigenvalues of $\mathcal{Q}_N$ $(p = 0)$ and $\mathcal{Q}_{N,p}$, with $p = 1$?



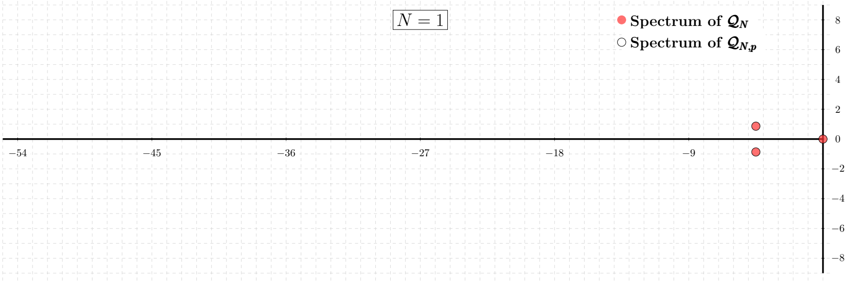
Spectrum of $\mathcal{Q}_{N,p}$ (numerical example)

Consider the mutation matrix

$$Q = \begin{pmatrix} -3 & 1 & 2 \\ 2 & -3 & 1 \\ 1 & 2 & -3 \end{pmatrix}$$

The eigenvalues of Q are $\lambda_0 = 0$ and $\lambda_1 = \overline{\lambda_2} = -\frac{9}{2} + \mathrm{i}\frac{\sqrt{3}}{2}$

➜ Eigenvalues of $\mathcal{Q}_N$ ($p = 0$) and $\mathcal{Q}_{N,p}$, with $p = 1$?



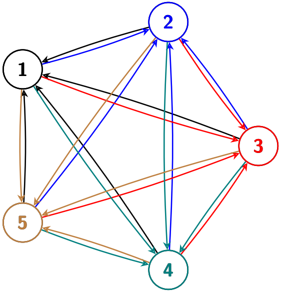
Parent independent mutation (PIM)

"the mutation rates only depend on the type of the new individual"

Etheridge (2011)

$$Q := \begin{pmatrix} -|\boldsymbol{\mu}| + \mu_1 & \mu_2 & \dots & \mu_K \\ \mu_1 & -|\boldsymbol{\mu}| + \mu_2 & \dots & \mu_K \\ \vdots & \vdots & \ddots & \vdots \\ \mu_1 & \mu_2 & \dots & -|\boldsymbol{\mu}| + \mu_K \end{pmatrix},$$

for some vector $\boldsymbol{\mu} \in (\mathbb{R}_+^*)^K$



Generator of the neutral Moran model with PIM:

$$\mathcal{Q}_{N,p}[\eta, \eta - \mathbf{e}_i + \mathbf{e}_j] = \eta(i) \left(\boldsymbol{\mu_j} + p\frac{\eta(j)}{N} \right)$$

Reversibility of the neutral multi-allelic Moran process with $p > 0$

if and only if the mutation rate matrix is **parent independent**.

Spectral decomposition

- eigenvalues of $\mathcal{Q}_{N,p}$:

$$\lambda_{n,p} = -|\boldsymbol{\mu}|n - \frac{p}{N}n(n-1), \text{ of multiplicity } \binom{K+n-2}{n},$$

for $n \in \{0, 1, \ldots, N\}$

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➥ Cutoff phenomenon!

Corollary

Assume that **(AD)**, **(IC)** and **(UC)** are verified. Then,

- Almost sure convergence:

$$m\left(\eta_T^{(N)}\right)(\phi) \xrightarrow[N\to\infty]{\text{a.s.}} \mu_T(\phi)$$

- Convergence of the mean empirical measure:

$$\sup_{t\geq 0} \left\| \bar{m}\left(\eta_t^{(N)}\right) - \mu_t \right\|_{\mathrm{TV}} \leq \frac{C}{N}, \text{ where } \bar{m}(\eta_t^{(N)}) := \sum_{x\in E} \mathbb{E}\left[\frac{\eta_t^{(N)}(x)}{N}\right] \delta_x$$

- Moreover, if the initial distribution of the N particles is exchangeable, then

$$\sup_{t\geq 0} \left\| \mathrm{Law}(\xi_t^{(i)}) - \mu_t \right\|_{\mathrm{TV}} \leq \frac{C}{N}$$