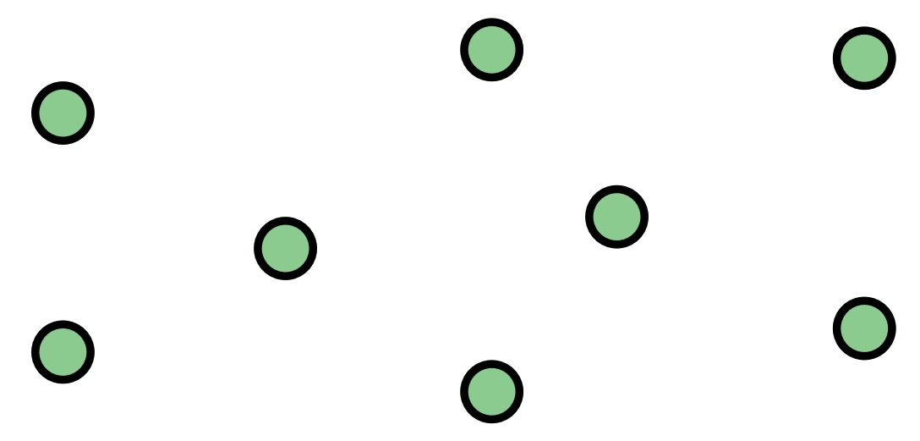
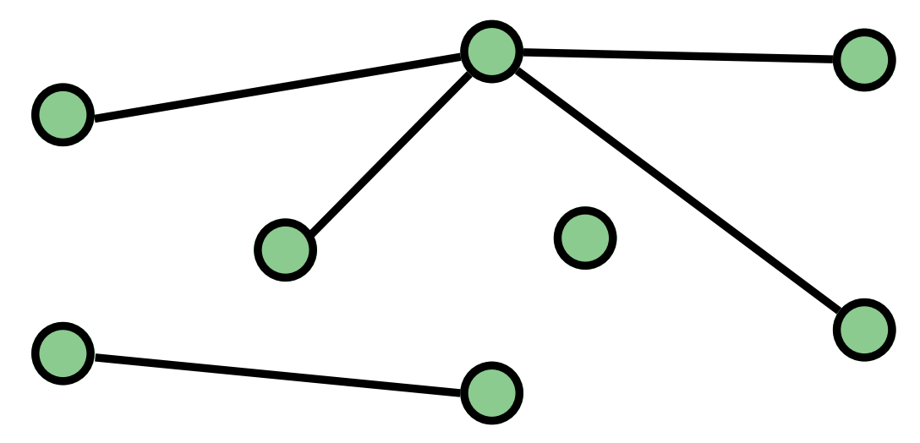
A modern look into the Erdős–Rényi random process

Erdös-Rényi random graph

1. Start with *n* isolated vertices



2. For each pair of vertices, add an edge with probability *p*, independently



$$\mathrm{G}(n,p)$$

number of vertices

probability that an edge exists

What does it model?

Epidemics

spreading of a disease according to connectivity among invidiuals

Networks

connection among websites according to links

Percolation

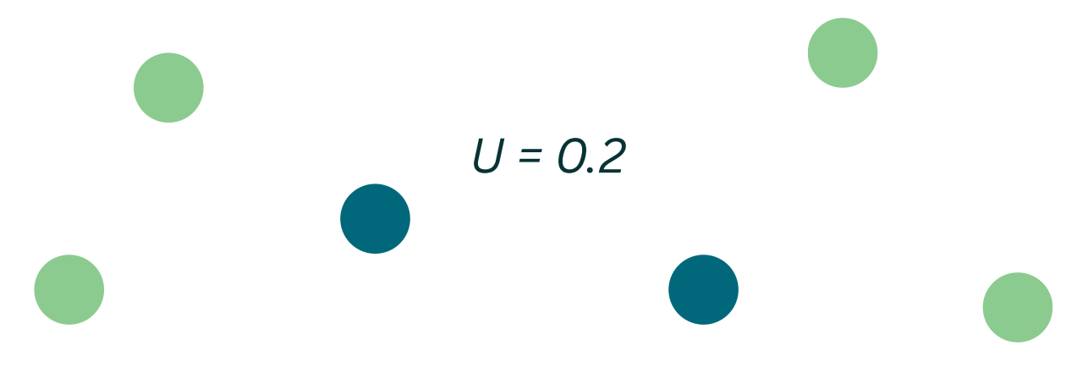
possibility of a fluid to flow through a material

this model is limited for applications
- there is no geometry
- edges exist with the same probability and independently

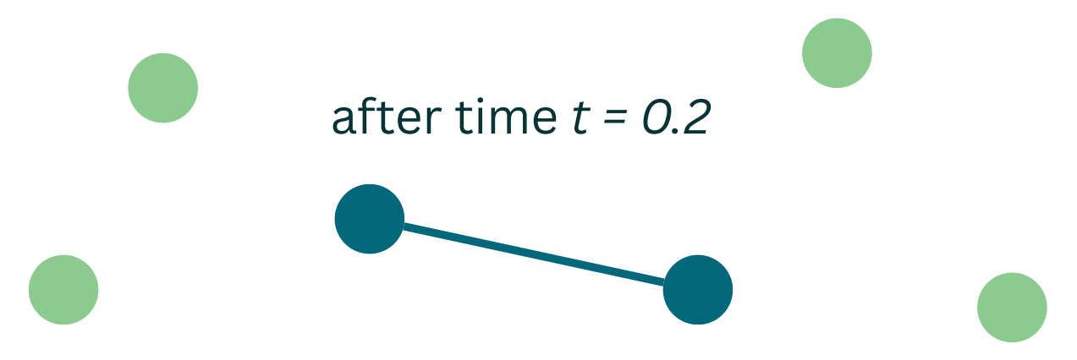
For *n* fixed, there is a natural way to embed every Erdös-Rényi random graph in the same space

1. Start with *n* isolated vertices
2. For each pair of vertices

pick a random number *U* in *[0,1]*



the edge exists after time *U*



now, instead of one random graph, we have an Erdös-Rényi random graph with probability *t* for every *t* in *[0,1]*

$$(\mathrm{G}(n,t))_{t\in[0,1]}$$

The phase transition

We will study the behavior of $\left(\mathrm{G}\left(n,\frac{c}{n}\right)\right)_{c\geq 0}$

THE MEAN NUMBER OF EDGES FROM ANY VERTEX IS C

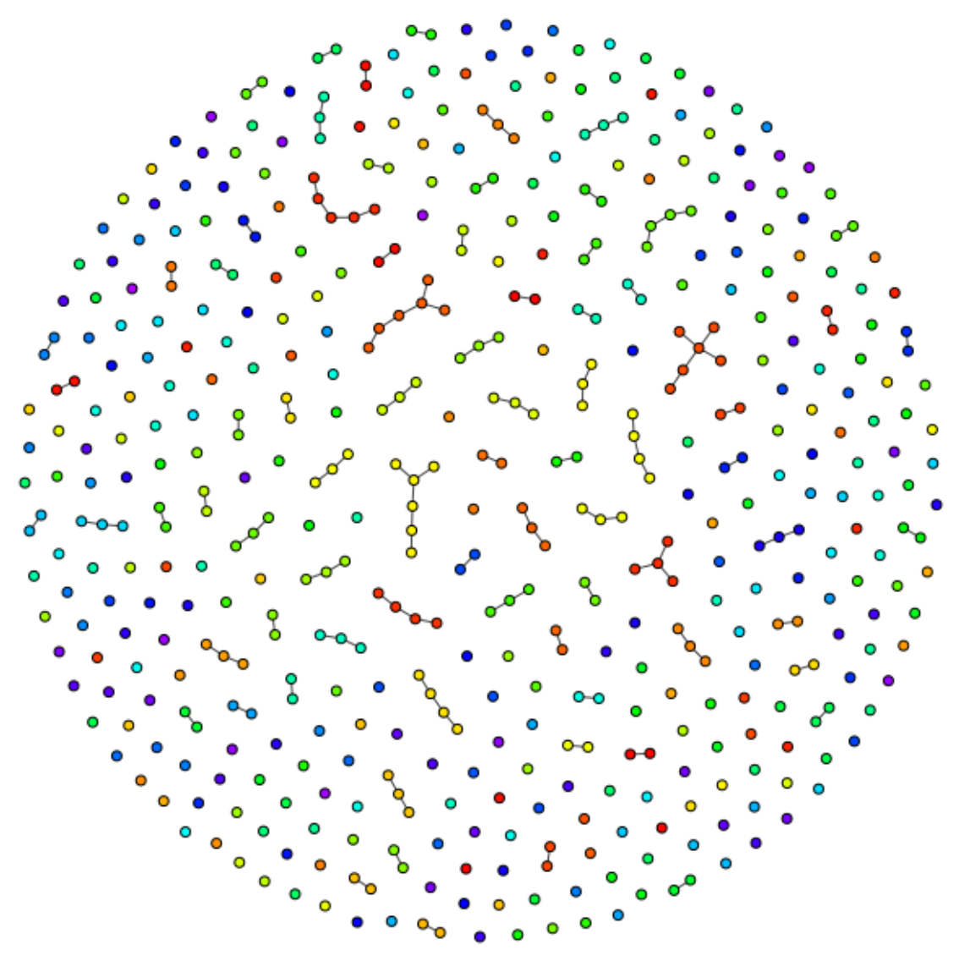
1. sub-critical

$0\leq c<1$

few edges

connected components are small

infection stays in small groups: there is no epidemic



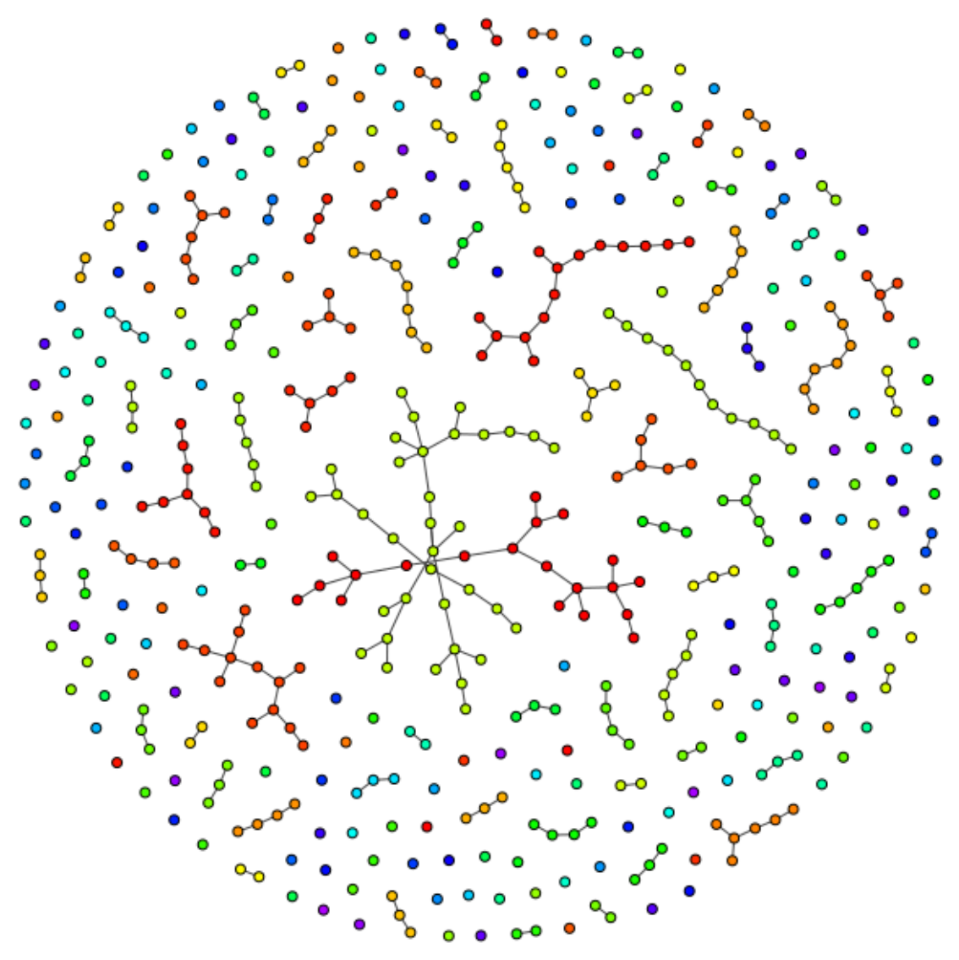
2. critical

$c=1$

average degree of a vertex is 1

many connected components of medium size

a giant component is about to appear



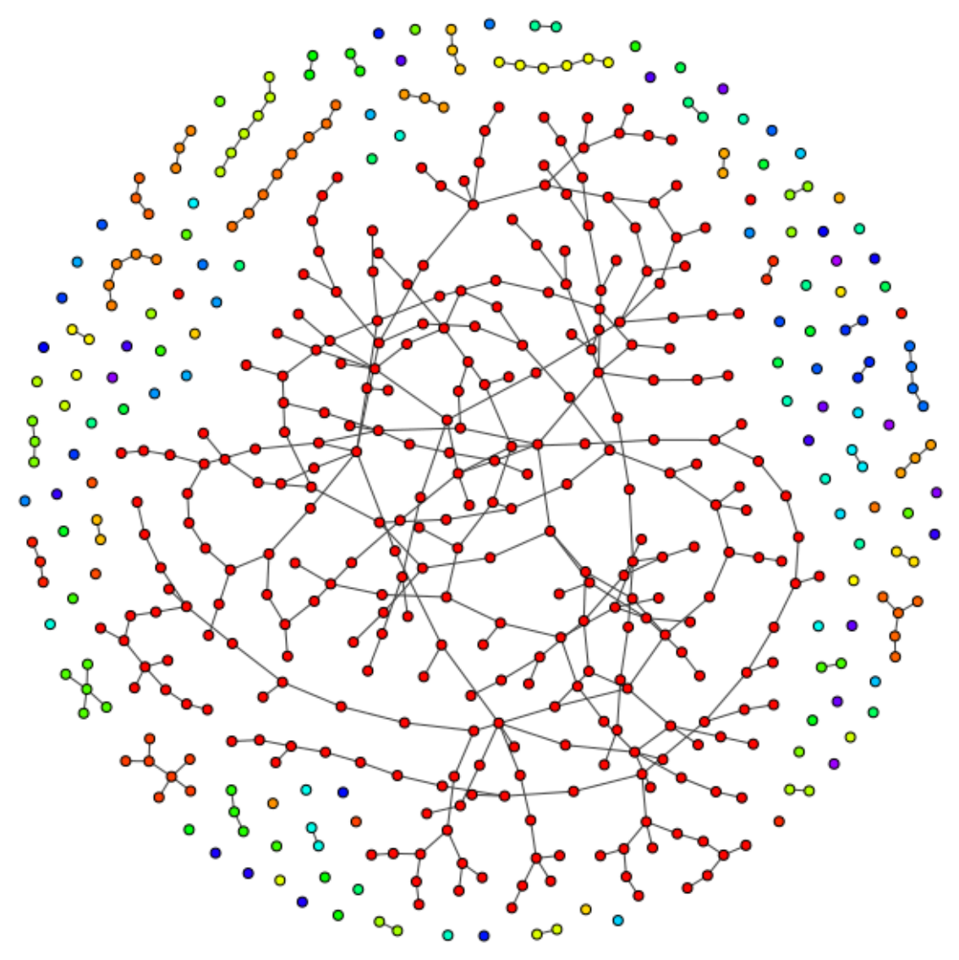
3. super-critical

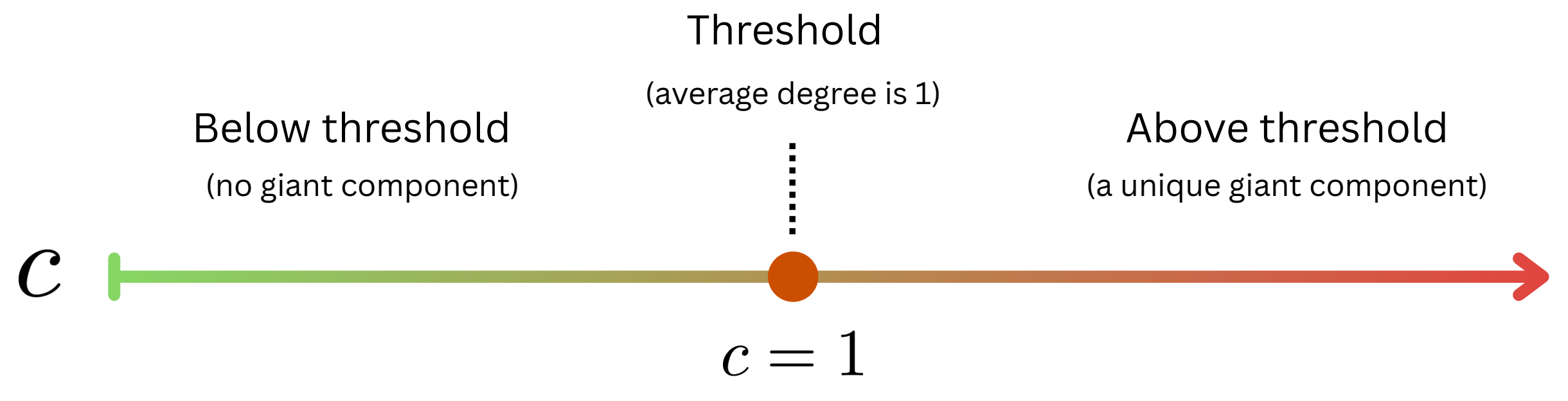
$c>1$

average degree of a vertex is finite but larger than 1

there is a unique giant component

infection can propagate trough larger groupes: epidemic





A REALIZATION OF THE PROCESS FROM SUB-CRITICAL TO CONNECTED (BY @EPFLNOOC)

(LOOK TO WHAT HAPPENS WHEN THE PARAMETER GOES THROUGH 1.0, AROUND SECOND 40)

Number of connected components

$K\left(\frac{c}{n}\right) \longrightarrow$ number of connected components in $G\left(n,\frac{c}{n}\right)$

Puhalskii (2005):

Fluid limit: for every $0\leq c<1$

$$\frac{1}{n}K\left(\frac{c}{n}\right)\xrightarrow[n\to\infty]{\mathbb{P}}1-\frac{c}{2}$$

Diffusion limit: for every $0\leq c<1$

$$\sqrt{n}\left(\frac{1}{n}K\left(\frac{c}{n}\right)-\left(1-\frac{c}{2}\right)\right)\xrightarrow[n\to\infty]{\text{law}}\mathrm{N}\left(0,\frac{c}{2}\right)$$

GAUSSIAN DISTRIBUTION WITH MEAN 0 AND VARIANCE C/2 (THE SAME DISTRIBUTION AS IN THE CENTRAL LIMIT THEOREM)

Size of the giant component

$L\left(\frac{c}{n}\right) \longrightarrow$ size of the largest connected components in $G\left(n,\frac{c}{n}\right)$

Erdös and Rényi (1959), Stepanov (1970):

Fluid limit: for every $0\leq c<1$

$$\frac{1}{n}L\left(\frac{c}{n}\right)\xrightarrow[n\to\infty]{\mathbb{P}}\rho(c)$$

solution of $1-\mathrm{e}^{-cx}-x=0$

Diffusion limit: for every $0\leq c<1$

$$\sqrt{n}\left(\frac{1}{n}L\left(\frac{c}{n}\right)-\rho(c)\right)\xrightarrow[n\to\infty]{\text{law}}\mathrm{N}\left(0,\sigma^2(c)\right)$$

where $\sigma^2(c)=\dfrac{\rho(c)(1-\rho(c)}{(1-c(1-\rho(c)))^2}$

what can we say about the limit of the processes of fluctuations

$$\left(\sqrt{n}\left(\frac{L(c/n)}{n}-\rho(c)\right),c>1\right)\quad\text{and}\quad\left(\sqrt{n}\left(\frac{1}{n}K\left(\frac{c}{n}\right)-\left(1-\frac{c}{2}\right)\right),c<1\right)\ ?$$

what is the limit of the covariances

$$\mathrm{Cov}\left(L\left(\frac{c_1}{n}\right),L\left(\frac{c_2}{n}\right)\right)\quad\text{and}\quad\mathrm{Cov}\left(K\left(\frac{c_1}{n}\right),K\left(\frac{c_2}{n}\right)\right)\ ?$$

Dynamical fluctuations

C. (2024+)

$$\left(\sqrt{n}\left(\frac{K(c/n)}{n}-\left(1-\frac{c}{2}\right)\right),c<1\right)$$

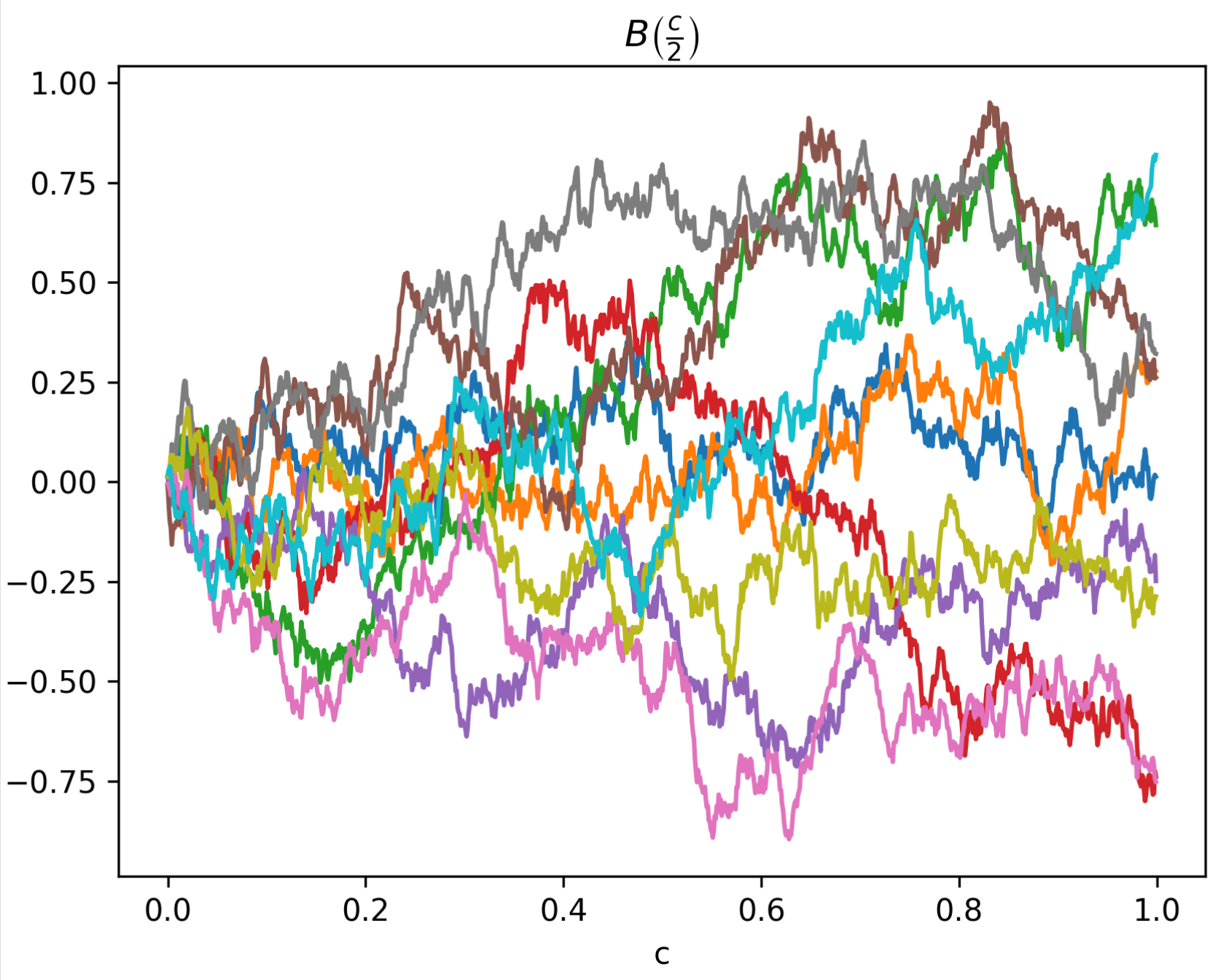
converges in law to

$$\left(B\left(\frac{c}{2}\right),c<1\right)$$

standard Brownian motion

$$\frac{1}{n}\mathrm{Cov}\left(K\left(\frac{c_1}{2}\right),K\left(\frac{c_2}{n}\right)\right)\xrightarrow[n\to\infty]{}\frac{c_1}{2}$$

for $0\leq c_1\leq c_2<1$



Remarks
- the result is actually stated for more general (inhomogeneous) random graph processes
- the proof consists on the study of a martingale related to the multiplicative coalescent dynamics

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Enriquez, Faraud and Lemaire (2023)
C., Lemaire and Limic (2024+)

$$\left(\sqrt{n}\left(\frac{L(c/n)}{n}-\rho(c)\right),c>1\right)$$

converges in law to

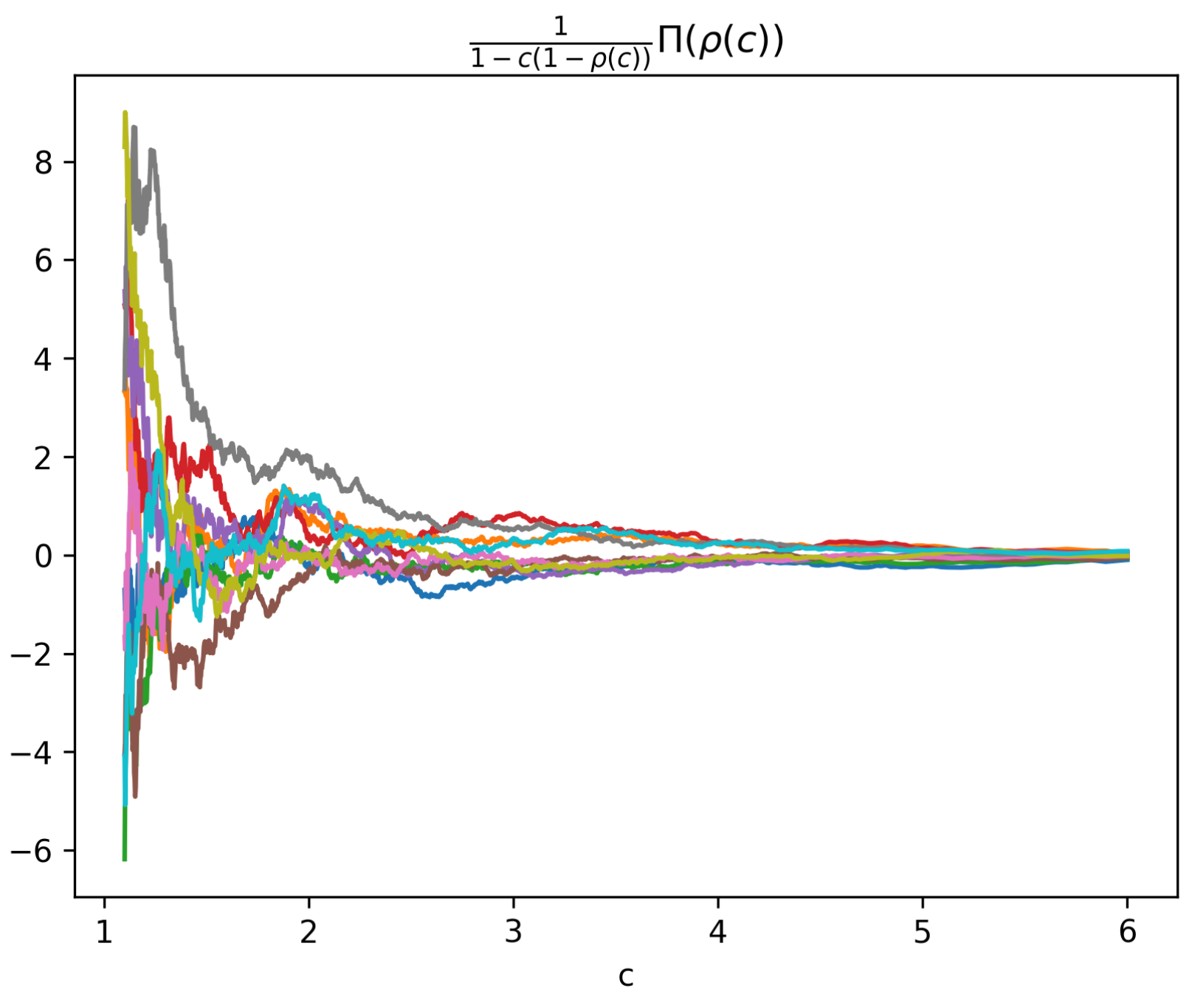
$$\left(\frac{1}{1-c(1-\rho(c))}\Pi(\rho(c)),c>1\right)$$

STANDARD BROWNIAN BRIDGE

$\frac{1}{n}\mathrm{Cov}\left(L\left(\frac{c_1}{n}\right),L\left(\frac{c_2}{n}\right)\right)$ converges to

$$\frac{\rho(c_1)(1-\rho(c_2))}{(1-c_1(1-\rho(c_1)))(1-c_2(1-\rho(c_2)))}$$

for $0\leq c_1\leq c_2<1$



Remarks
- our proof is based on the study of an encoding process called *simultaneous breadth-first walk*, introduced by Limic (2019)
- in the same article, we also study the fluctuations in the barely super-critical regime, when the time is $\frac{1+t\cdot\epsilon_n}{n}$ with $\epsilon_n\to 0$ and $\sqrt[3]{n}\cdot\epsilon_n\to\infty$
- by using the same techniques, Clancy Jr. (2025), proves an analogous result for a class of inhomogeneous random graph processes

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